

Lecture 11: Surfaces w/ $K=1$ & Elliptic Surfaces

Lemma 1. $X =$ non-ruled minimal surface

(a) If $K^2 > 0$, $\exists n_0 \in \mathbb{N}$ st $\varphi_{nk}: X \xrightarrow{|nK_x|} \mathbb{P}^N$
maps X birationally onto its image, for $n \geq n_0$.

(b) $K^2 = 0$, $P_g \geq 2$, write $rK_x = Z + M$,
fixed mobile
part part

Then $K_x \cdot Z = K_x \cdot M = Z^2 = Z \cdot M = M^2 = 0$.

pf: (a) H : very ample divisor.

RR $h^0(nK_x - H) + h^0(H + (1-n)K_x) \rightarrow \infty$ as $n \rightarrow \infty$
!" $K^2 > 0$

X non-ruled $\Rightarrow K_x \cdot H > 0$
 $\Rightarrow (H + (1-n)K_x) \cdot H < 0$ as $n \gg 0$
 $\Rightarrow h^0(H + (1-n)K_x) = 0$, $n \gg 0$
 $\Rightarrow h^0(nK_x - H) > 0$, $n \gg 0$

Take $E \in |nK_x - H|$, then $E + H \in |nK_x|$.

Since H is very ample, base point of $|nK_x| \subseteq E$.

$$\begin{array}{ccc} \therefore X & \xrightarrow{|nK_x|} & \mathbb{P}^\star \\ \cup & & \cup \\ X/E & \xrightarrow{\cong} & \mathbb{P}_{nK_x}(X/E) \end{array}$$

(b) $rK_x^2 = K_x \cdot M + K_x \cdot Z$
 $\forall \circ: M$ mobile $\forall \circ$ Otherwise \exists component $Z' \subseteq Z$
 \circ \circ s.t. $Z'^2 < 0$, $K_x \cdot Z' < 0$
 \circ \circ contradicts to X minimal.

$$\Rightarrow \underline{K_X \cdot M} = K_X \cdot Z = 0$$

$$\swarrow \begin{matrix} \because M \cdot Z \geq 0 \\ M^2 \geq 0 \end{matrix}$$

$$M^2 = M \cdot Z = 0$$

$$0 = (rK_X)^2 = (Z+M)^2 = Z^2 + \cancel{2Z \cdot M} + \cancel{M^2} \Rightarrow Z^2 = 0$$

Proposition 1: X minimal surface w/ $\kappa=1$

$$\Rightarrow (a) K_X^2 = 0$$

(b) $\exists X \xrightarrow{P} B$, generic fibre elliptic curve.

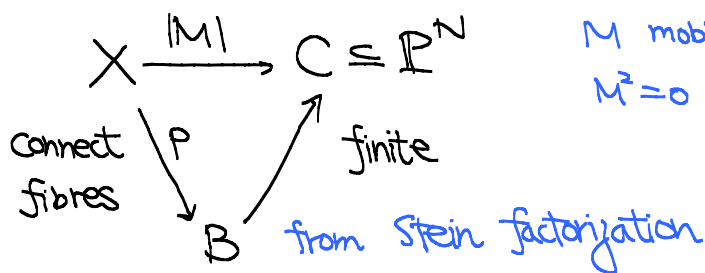
X is called an elliptic surface

pf: (a) Lemma 1 (a) $\Rightarrow K_X^2 \leq 0$

$K_X^2 < 0 + X$ minimal $\Rightarrow X$ is ruled $\therefore \kappa = -\infty$

Therefore $K_X^2 = 0$

(b) Lemma 1 (b) $\Rightarrow rK_X = \underbrace{Z}_{\text{fixed part}} + \underbrace{M}_{\text{mobile part}}$, $M^2 = Z \cdot M = 0$



M mobile \Rightarrow image not a point
 $M^2=0 \Rightarrow$ image not a surface

F : generic fibre

$$K_X \cdot F = K_X \cdot M = 0$$

$$\therefore 2-2g_F = K_X \cdot F + F^2 = 0$$

$$\text{or } g_F = 1$$

\uparrow
 M is a sum of
 fibre of P .

Not all elliptic surface has $\kappa=1$.

Proposition 2: $X =$ minimal elliptic fibration

$$\begin{array}{ccc} X & \xrightarrow{\pi} & B \\ \subseteq & & \in \\ F_b & \longrightarrow & b \end{array}$$

$$\implies (a) \quad K_X^2 = 0$$

(b) X either ruled over an elliptic curve
 or surface w/ $K=0$
 or surface w/ $K=1$

(c) If $K=1$, then $\exists d \in \mathbb{N}, b_i \in B, n_i \in \mathbb{N}$
 s.t $\sum_i n_i F_{b_i} \in |dK_X|$

For $r \gg 0$, $|rdK_X|$ has no base points

$$\text{thus } \begin{array}{ccc} X & \xrightarrow{|rdK_X|} & \mathbb{P}^N \\ & \searrow \pi & \nearrow \\ & B & \end{array}$$

pf: (a) If X is ruled, $X \xrightarrow{P} C$

the elliptic fibres can't be contracted by $P \Rightarrow g_C = 1$

$$\therefore K_X^2 = 8(1 - g_C) \geq 0$$

$\implies K_X^2 \geq 0$ for any minimal elliptic surface X .

If $|nK_X| \neq \emptyset$ for some $n \in \mathbb{N}$

$D \in |nK_X|$
 $D \cdot F = 0 \implies$ components of D are
 (F) base point free Contained in fibres

$$\therefore D^2 \leq 0$$

Thus $D = \sum_i n_i F_{b_i}$ for some $n_i \in \mathbb{N}$, $b_i \in \mathcal{B}$
 X : minimal ruled $\Rightarrow X \cong \mathbb{P}(E)$
 $\Rightarrow K_X^2 = 0$

If X not ruled, minimal surface.

R.R. $h^0(nK_X) + h^0((1-n)K_X) \rightarrow \infty$ if $K_X^2 > 0, n \gg 0$
 $\parallel \begin{matrix} \because K_X \cdot H > 0 \\ 0 & (1-n)K_X \cdot H < 0 \end{matrix}$

$\therefore h^0(nK_X) > 0 \Rightarrow K_X^2 = 0 \xrightarrow{K_X^2 > 0} \times$

Therefore, $K_X^2 = 0$ if X is minimal elliptic.

(b) $K_X^2 = 0, K_X \cdot F = 0 \Rightarrow \phi_{|nK_X|}$ contracts elliptic fibres
 $\Rightarrow \underline{K = -\infty, 0, 1}$
 \Downarrow
 X is ruled

(c) $K=1, \exists r \in \mathbb{N}$ s.t. $P_r > 1$

$D \in |rK_X|, D \sim \sum_i r_i F_{b_i}, r_i \in \mathbb{Q}, b_i \in \mathcal{B}, X \xrightarrow{P} B \ni b_i$
 elliptic fibration

then $dK_X \sim \sum_i n_i F_{b_i}, n_i \in \mathbb{Z}$, for suitably divisible d .
 \parallel
 $p^* \left(\underbrace{\sum_i n_i [b_i]}_A \right)$

For $r \gg 0, |rA|$ is very ample $\rightsquigarrow j: B \xrightarrow{|rA|} \mathbb{P}^N$

In particular, $p^*|rA| \subseteq |rdK_X|$
 is base point free.

Examples of elliptic surface $w/ K \neq 1$

1. $E \times \mathbb{P}^1 / G$, E : elliptic curve
 G : finite group action on E, \mathbb{P}^1
 both ruled & elliptic.

2. Bi-elliptic surfaces $E \times F / G$, E, F elliptic curve

3. $0 \rightarrow E \rightarrow \boxed{A \rightarrow A/E} \rightarrow 0$
 elliptic curve abelian surface elliptic fibration

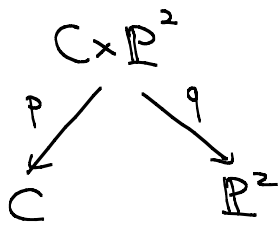
4. Elliptic K3 surfaces

5. Enriques surfaces always admit elliptic fibrations.

6. Rational elliptic surfaces.

Example of elliptic surface $w/ K=1$.

C : smooth curve $w/$ base point free linear system $|D|$



$$H^0(C \times \mathbb{P}^2, p^* \mathcal{O}_C(D) \otimes q^* \mathcal{O}_{\mathbb{P}^2}(3))$$

\downarrow
 S generic

$$X = \{s=0\} \subseteq C \times \mathbb{P}^2$$



$$K_X = p^*(K_C + D)$$

$$H^0(X, K_X) \cong H^0(C, K_C + D)$$

dimension $2g-2 + \deg D - g + 1$

If $\deg D \geq 1 - g + 2$, then $K(X) = 1$