# Lecture 11: Surfaces w/ $\kappa=1$ & Elliptic Surfaces

**Lemma 1.** $X$ = non-ruled minimal surface

(a) If $K^2>0$, $\exists\, n_0\in\mathbb{N}$ s.t. $\varphi_{nK}: X \xrightarrow{|nK_X|} \mathbb{P}^N$ maps $X$ birationally onto it image, for $n\geq n_0$.

(b) $K^2=0$, $P_r\geq 2$, write $rK_X = \underset{\text{fixed part}}{Z} + \underset{\text{mobile part}}{M}$.

Then $K_X.Z = K_X.M = Z^2 = Z.M = M^2 = 0$.

pf: (a) $H$: very ample divisor.

RR $\quad h^0(nK_X-H) + h^0(H+(1-n)K_X) \longrightarrow \infty$ as $n\to\infty$
$\because K^2>0$

$X$ non-ruled $\Rightarrow K_X.H>0$

$\Rightarrow (H+(1-n)K_X).H<0$ as $n\gg 0$

$\Rightarrow h^0(H+(1-n)K_X)=0$, $n\gg 0$

$\Rightarrow h^0(nK_X-H)>0$, $n\gg 0$

Take $E\in|nK_X-H|$, then $E+H\in|nK_X|$.

Since $H$ is very ample, base point of $|nK_X|\subseteq E$.

$$\therefore \begin{array}{ccc} X & \overset{|nK_X|}{\dashrightarrow} & \mathbb{P}^* \\ \cup & & \cup \\ X\setminus E & \xrightarrow{\simeq} & \Phi_{nK_X}(X\setminus E) \end{array}$$

(b) $rK_X^2 = \underset{\substack{\vee\!\!| \\ 0}}{K_X.M} + \underset{\substack{\vee\!\!| \\ 0}}{K_X.Z}$ $\quad \because M$ mobile

Otherwise $\exists$ component $Z'\subseteq Z$ s.t. $Z'^2<0$, $K_X.Z'<0$ contradicts to $X$ minimal.

$$\Longrightarrow \underline{K_X.M} = K_X.Z = 0$$

$\because M.Z \geqslant 0$, $M^2 \geqslant 0$

$M^2 = M.Z = 0$

$$0 = (rK_X)^2 = (Z+M)^2 = Z^2 + \cancel{2Z.M}^{\,0} + \cancel{M^2}^{\,0} \Longrightarrow Z^2 = 0.$$

**Proposition 1**: $X$ minimal surface w/ $\kappa = 1$

$\Longrightarrow$ (a) $K_X^2 = 0$

(b) $\exists\ X \overset{p}{\twoheadrightarrow} B$, generic fibre elliptic curve.

$X$ is called an <u>elliptic surface</u>

pf: (a) Lemma 1 (a) $\Longrightarrow K_X^2 \leq 0$

$K_X^2 < 0$ + $X$ minimal $\Longrightarrow$ $X$ is ruled $\therefore \kappa = -\infty$

Therefore $K_X^2 = 0$

(b) Lemma 1 (b) $\Longrightarrow$ $rK_X = \underset{\text{fixed part}}{Z} + \underset{\text{mobile part}}{M}$, $M^2 = Z.M = 0$

$$\begin{tikzcd} X \arrow[r, "|M|"] \arrow[dr, "p", "\text{connect fibres}"'] & C \subseteq \mathbb{P}^N \\ & B \arrow[u, "\text{finite}"'] \end{tikzcd}$$

from Stein factorization

$M$ mobile $\Rightarrow$ image not a point

$M^2 = 0 \Rightarrow$ image not a surface

$F$: generic fibre

$K_X.F = K_X.M = 0$ $\therefore 2 - 2g_F = K_X.F + F^2 = 0$

or $g_F = 1$

$M$ is a sum of fibre of $p$.

Not all elliptic surface has $\kappa = 1$.

Proposition 2: $X$ = minimal elliptic fibration

$$\begin{array}{ccc} X & \xrightarrow{\pi} & B \\ \cup & & \in \\ F_b & \longrightarrow & b \end{array}$$

$\Longrightarrow$ (a) $K_X^2 = 0$

(b) $X$ either ruled over an elliptic curve

or surface w/ $\kappa = 0$

or surface w/ $\kappa = 1$

(c) If $\kappa = 1$, then $\exists\, d \in \mathbb{N},\ b_i \in B,\ n_i \in \mathbb{N}$

s.t. $\sum_i n_i F_{b_i} \in |dK_X|$

For $r \gg 0$, $|rdK_X|$ has no base points

thus
$$\begin{tikzcd} X \arrow[r, "|rdK_X|"] \arrow[dr, "\pi"'] & \mathbb{P}^N \\ & B \arrow[u, hook] \end{tikzcd}$$
with $\cong$ written above the arrow $\pi$.

pf: (a) If $X$ is ruled, $X \xrightarrow{p} C$

the elliptic fibres can't be contracted by $p \Rightarrow g_C \leq 1$

$\therefore K_X^2 = 8(1-g_C) \geq 0$

$\Longrightarrow$ $K_X^2 \geq 0$ for any minimal elliptic surface $X$.

If $|nK_X| \neq \phi$ for some $n \in \mathbb{N}$

$\in$
$D$ $\quad D.F = 0 \Longrightarrow$ components of $D$ are

$|F|$ base point free $\quad$ contained in fibres

$\therefore D^2 \leq 0$

Thus $D = \sum_i n_i F_{b_i}$ for some $n_i \in \mathbb{N}$, $b_i \in B$

$X$: minimal ruled $\Rightarrow$ $X \cong \mathbb{P}(E)$

$\Longrightarrow K_X^2 = 0$

If $X$ not ruled, minimal surface.

R.R. $h^0(nK_X) + h^0((1-n)K_X) \longrightarrow \infty$ if $K_X^2 > 0$, $n \gg 0$

$$\underset{0}{\|} \quad \because K_X.H > 0, \ (1-n)K_X.H < 0$$

$\therefore h^0(nK_X) > 0 \Longrightarrow K_X^2 = 0 \ \cancel{\longrightarrow}$ $K_X^2 > 0$

Therefore, $K_X^2 = 0$ if $X$ is minimal elliptic.

(b) $K_X^2 = 0$, $K_X.F = 0 \Longrightarrow \varphi_{|nK_X|}$ contracts elliptic fibres

$\Longrightarrow \underline{\kappa = -\infty}, 0, 1$

$\Downarrow$

$X$ is ruled

(c) $\kappa = 1$, $\exists r \in \mathbb{N}$ s.t $P_r > 1$

$D \in |rK_X|$, $D \sim \sum_i r_i F_{b_i}$, $r_i \in \mathbb{Q}$, $b_i \in B$, $X \xrightarrow{p} B \ni b_i$ elliptic fibration

then $dK_X \sim \sum_i n_i F_{b_i}$, $n_i \in \mathbb{Z}$, for suitably divisible $d$.

$$\| \quad p^*\big(\underbrace{\textstyle\sum_i n_i[b_i]}_{A}\big)$$

For $r \gg 0$, $|rA|$ is very ample $\rightsquigarrow j: B \xhookrightarrow{|rA|} \mathbb{P}^N$

In particular, $p^*|rA| \subseteq |rdK_X|$

is base point free.

## Examples of elliptic surface w/ $\kappa \neq 1$

1. $E \times \mathbb{P}^1 / G$. $\quad E$: elliptic curve
   $G$: finite group action on $E$, $\mathbb{P}^1$
   both ruled & elliptic.

2. Bi-elliptic surfaces $E \times F / G$. $E, F$ elliptic curve

3. $$0 \to \underset{\text{elliptic curve}}{E} \to \boxed{\underset{\text{abelian surface}}{A} \to A/E} \to 0$$
   elliptic fibration

4. Elliptic K3 surfaces

5. Enrique surfaces always admit elliptic fibrations.

6. Rational elliptic surfaces.

## Example of elliptic surface w/ $\kappa = 1$.

$C$: smooth curve w/ base point free linear system $|D|$

$$\begin{tikzcd} & C \times \mathbb{P}^2 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C & & \mathbb{P}^2 \end{tikzcd}$$

$$H^0(C \times \mathbb{P}^2, p^* \mathcal{O}_C(D) \otimes q^* \mathcal{O}_{\mathbb{P}^2}(3))$$
$$\ni$$
$$s \quad \text{generic}$$

$$\begin{tikzcd} X = \{s = 0\} \subseteq C \times \mathbb{P}^2 \arrow[dr, "\text{elliptic fibration}"'] \arrow[d, "q"] \\ \mathbb{P}^2 \end{tikzcd}$$

$K_X = p^*(K_C + D)$ $\qquad H^0(X, K_X) \cong H^0(C, K_C + D)$

dimension $2g - 2 + \deg D - g + 1$

If $\deg D \geq 1 - g + 2$, then $\kappa(X) = 1$